Int J Theor Phys (2008) 47: 1865–1876
DOI 10.1007/s10773-007-9631-8

Solutions of Two-Mode Bosonic and Transformed Hamiltonians

Sema Bilge Ocak · Özlem Yeşiltaş · Bengü Demircioğlu

Received: 30 July 2007 / Accepted: 5 December 2007 / Published online: 13 December 2007

Abstract We have constructed the quasi-exactly-solvable two-mode bosonic realization of $SU(2)$. Two-mode boson Hamiltonian is defined through a differential equation which is solved by quantum Hamilton-Jacobi formalism. The squeezed states of two-mode boson systems are characterized through canonical transformation. The illustrated concept of squeezed boson systems has been applied two-mode bosonic Hamiltonian which is a squeezed one and is determined through a differential equation. This differential equation is solved and energy eigenvalues are found approximately.

Keywords Squeezed · QHJ-formalism · Boson

1 Introduction

In quantum mechanics, investigating the exact solutions (ES) of Schrödinger equation is one of the most popular field. Nevertheless, there are only a few problems of exactly solvable systems such that various methods are explored in theoretical physics and mathematics for obtaining the solutions [1–9]. Beside the ES problems, quasi-exactly solvable (QES) systems have been attracted great interest [10–15]. QES problems have been extensively studied in an algebraic approach [15–21]. On the other hand, quantum Hamilton-Jacobi (QHJ) formalism has generated much interest [22, 24, 25]. The application of QHJ to eigenvalues has been explored in great detail by Bhalla, Kapoor and collaborators [26–33]. QES systems have been also studied within QHJ approach [31].

As an another research field, squeezed states play an important role in quantum optics and condensed matter [34–37] which are now also considered for boson systems [38] and it can be investigated using the solutions obtained from Hamiltonian with quartic potential. Squeezed states of boson systems are defined through canonical transformation. Such a

S.B. Ocak (✉) · B. Demircioğlu
Faculty of Science, Ankara University, 06100, Ankara, Turkey
e-mail: semamuzo@yahoo.com

Ö. Yeşiltaş
Department of Physics, Faculty of Arts and Sciences, Gazi University, 06500, Ankara, Turkey

transformation results from parametric interaction with a strong classical pump for bosonic squeezing. The simplest nonlinear interaction Hamiltonian with quadratic terms of annihilation (a) and creation (a^+) boson operators and with interaction parameter χ

$$H_I = \hbar(\chi a^2 + \chi^*(a^+)^2), \tag{1}$$

gives a unitary transformation $U = \exp(itH_I/\hbar)$, from the squeezed states of light evolve either from the coherent states or equivalently the vacuum state. It also determines the general form of the well known unitary transformation that leads to diagonalization of any Hamiltonian containing quadratic terms.

In the bosonic Hamiltonian, it is more convenient to use the bosonic representations of the $SU(2)$ and $SU(1)$ generators and determine the conditions of quasi-exact solvability. The single boson realization of the $SU(1,1)$ algebra has been studied in [39, 40]. In this work we follow a different strategy to obtain the condition for QES of the two-bosonic systems.

One may also readily generalize the current approach for two-level models involving linear or quadratic interactions with a single boson (or canonical quantum mode) to the corresponding case of linear or bilinear interactions involving several distinct bosons or modes. For the linear models involving only displacements this is essentially trivial. However, for models involving squeezing, in the case of n boson or modes the various bilinear products of operators $a_i^+ a_j^+, a_i a_j$ and $a_i^+ a_j,\ j,\ i = 1, 2, 3, \ldots, n$ now form a realization of higher symplectic algebra $Sp(2n, R)$. Two-mode squeezed states are associated with a unitary representation of group $Sp(4, R)$. This $Sp(4, R)$ algebra has various subalgebras corresponding to different sorts of linear pairing terms. For example, whereas the single-mode pairing operators $K_+^i = (1/2)(a_i^+)^2$; $K_-^i = (1/2)(a_i)^2$; $K_0^i = (1/2)(a_i^+)a_i + (1/4)$ for $i = 1, 2$ correspond to the so-called $(1/4, 3/4)$ representation of $SU(1,1)$, the mixed pairing operators $L_+ = a^+b^+$, $L_- = ab$, $L_0 = (1/2)(a^+a + b^+b + 1)$ correspond to the discrete-series representation of $SU(1,1)$.

A convenient way [41] to construct a spectrum generating algebra for systems with a finite number of bound states is to introduce a set of boson creation and annihilation operators. We introduce two boson operators a_1 and a_2 which obey the usual commutation relation

$$J_+ = a_1^+ a_2, \qquad J_- = a_2^+ a_1 \quad \text{and} \quad J_3 = (a_1^+ a_1 - a_2^+ a_2)\frac{1}{2} \tag{2}$$

then we have the following commutators

$$[J_3, J_+] = J_+, \qquad [J_3, J_-] = -J_- \quad \text{and} \quad [J_+, J_-] = 2J_3. \tag{3}$$

The mixed pairing operators correspond to the angular momentum subalgebra $SU(2)$. If we consider a squeezed Hamiltonian, we can write

$$J_0 = \frac{1}{2}(a_1^+ a_1 + a_2^+ a_2), \qquad J_1 = \frac{1}{2}(a_1^+ a_2 + a_2^+ a_1), \tag{4a}$$

$$J_2 = \frac{1}{2i}(a_1^+ a_2 - a_2^+ a_1), \qquad J_3 = \frac{1}{2}(a_1^+ a_1 - a_2^+ a_2), \tag{4b}$$

$$K_1 = \frac{1}{2}(a_1^+ a_1 + a_1 a_1 - a_2^+ a_2^+ + a_2 a_2), \tag{4c}$$

$$K_2 = -\frac{i}{4}(a_1^+ a_1^+ - a_1 a_1 + a_2^+ a_2^+ - a_2 a_2), \qquad K_3 = \frac{1}{2}(a_1^+ a_2^+ + a_1 a_2), \tag{4d}$$

$$Q_1 = \frac{i}{2}(a_1^+ a_1^+ - a_1 a_1 - a_2^+ a_2^+ - a_2 a_2), \tag{4e}$$

$$Q_2 = -\frac{1}{4}(a_1^+ a_1^+ + a_1 a_1 + a_2^+ a_2^+ + a_2 a_2), \qquad Q_3 = \frac{i}{2}(a_1^+ a_2^+ - a_1 a_2). \tag{4f}$$

There are ten generators and one hundred commutation relations. Since they are antisymmetric, there are forty-five non-trivial commutators [42]. These ten generators form a closed set of commutation relations. It is indeed remarkable that this set of commutation relations is identical to that for $(3+2)$ dimensional Lorentz group which is commonly known as the $(3+2)$-dimensional de Sitter group.

Equations (4) can be transformed in the form of the one dimensional differential equations in the Bragmann-Fock space when the boson operators realized as

$$a_1 = \frac{d}{dx}, \qquad a_1^+ = x. \tag{5}$$

Therefore the differential realizations of the generators of K_+, K_-, J_+, J_-, J_0, L and Q_2 are given by

$$K_- = J_- = \frac{d}{dx}, \tag{6a}$$

$$K_+ = \left(x^2 \frac{d}{dx} - 2kx\right), \tag{6b}$$

$$J_+ = \left(x^2 \frac{d}{dx} + 2jx\right), \tag{6c}$$

$$J_0 = x\frac{d}{dx} - j, \tag{6d}$$

$$L = a_1^+ a_1 - a_2^+ a_2, \tag{6e}$$

$$\begin{aligned} Q_2 &= \frac{1}{2}J_0 K_+ - \frac{1}{2}J_0 K_- - \frac{1}{2}K_+ J_0 + \frac{1}{2}K_- J_0 \\ &= \frac{d^2}{dx^2}(x^3 - x^2) + \frac{d}{dx}(2x^2 - jx^2 + j + 2kx^2 + 1) - 2kx. \end{aligned} \tag{6f}$$

In this work we aim to solve two-mode boson Hamiltonian within the QHJ formalism. Using the solutions, the Hamiltonian is constructed through canonical transformation.The transformed Hamiltonian is transformed anew in the form of one-dimensional QES differential equation and approximate solutions are obtained. We analyze group of transformed Hamiltonian.

2 Quantum Hamilton Jacobi Theory

The QHJ formalism, another formulation of quantum mechanics, is closely related to the classical Hamilton-Jacobi theory. In the present formalism, the spectrum of one dimensional quantum mechanical system can be determined by the solution of the equation which is a special case of Riccati equation [22–25]:

$$-ip'(x, E) + p^2(x, E) = E - V(x), \tag{7}$$

from now on we use $\hbar = 2m = 1$. Here $p(x,E)$ is the quantum momentum function (QMF) which is defined

$$p = p(x,E) = \frac{\Psi'}{\Psi}. \tag{8}$$

As it seen from (8) $\Psi(x) \sim e^{i\int p(x,E)dx}$. When $\hbar \to 0$, the QHJ equation approaches to the classical Hamilton-Jacobi equation and QMF turns into the classical momentum function:

$$p(x,E) \overset{\hbar\to 0}{\to} p_{cl}(x,E) = p_{cl}, \tag{9}$$

where $p_{cl} = \sqrt{E - V(x)}$.

In the case of exactly solvable models, it is shown that one can use the quantization condition given below [31, 32]

$$\int p dq = nh. \tag{10}$$

More details can be found in [26–33].

3 The Model

We now consider the following bosonic operator

$$H = w_1 a_1^+ a_1 + w_2 a_2^+ a_2 + \alpha_1 a_1^+ a_2^+ + \alpha_2 a_1 a_2 - \frac{1}{2} a_1^2 a_2^2. \tag{11}$$

In terms of the generators of $SU(1,1)$ the Hamiltonian can be expressed as

$$H = (w_1 + w_2)\left(K_0 - \frac{1}{2}\right) + \frac{1}{2}(w_1 - w_2)L + \alpha_1 K_+ + \alpha_2 K_- - \frac{1}{2}K_-^2 \tag{12}$$

and it can be rewritten in the form of differential operator

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + (\alpha_2 + x(w_1 + w_2 + \alpha_1 x))\frac{d}{dx} - (2k\alpha_1 x + w_1(2k+1)) \tag{13}$$

which satisfies the following eigenvalue equation

$$HR(x) = ER(x). \tag{14}$$

The solution of (12) can be obtained by introducing the wave function

$$R(x) = e^{\int W(x)dx}\psi(x), \tag{15}$$

where the weight function $W(x)$ is given by

$$W(x) = -\alpha_2 - (w_1 + w_2)x - \alpha_1 x^2. \tag{16}$$

Using the wave function (15) and the Hamiltonian (13) in the eigenvalue equation (14) gives the Schrödinger equation with the potential

$$\begin{aligned} V(x) = {} & \frac{1}{2}(\alpha_2^2 - w_1(4k+3) - w_2) + (\alpha_2(w_1+w_2) - \alpha_1(2k+1))x \\ & + \frac{1}{2}(\alpha_1\alpha_2 + (w_1+w_2)^2)x^2 + \alpha_1(w_1+w_2)x^3 + \frac{\alpha_1^2}{2}x^4. \end{aligned} \tag{17}$$

In this section we now investigate the applicability of the QHJ formalism to solve one of the QES system. The effective potential of this system is written as

$$V_0(x) = A + Bx + Cx^2 + Dx^3 + Fx^4. \tag{18}$$

Using this potential in (7) leads to

$$-ip'(x, E) + p^2(x, E) - (E - A - Bx - Cx^2 - Dx^2 - Fx^4) = 0. \tag{19}$$

Now taking

$$p(x, E) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \tag{20}$$

in (19), we find

$$a_0 = \frac{-i}{2\sqrt{F}}\left(\frac{D^2}{4F} - C\right), \tag{21a}$$

$$a_1 = \pm i\frac{D}{2\sqrt{F}}, \tag{21b}$$

$$a_2 = \pm i\sqrt{F}, \tag{21c}$$

$$a_3 = 0 \tag{21d}$$

and the other potential parameter is found as

$$B = -\frac{D}{2F}\left(\frac{D^2}{4F} - C\right) - 2\sqrt{F}. \tag{22}$$

As it is seen from (21) and (22), a_2 and a_1 have two values. The correct one is chosen by the square integrability condition on the wave function. Therefore we require that $a_1 = i\frac{D}{2\sqrt{F}}$ and $a_2 = i\sqrt{F}$.

Now we can write the QMF as

$$p(x, E) = \sum_{k=1}^{n}\frac{-i}{x - x_k} + i\sqrt{F}x^2 + \frac{iD}{2\sqrt{F}}x - \frac{1}{2\sqrt{F}}\left(\frac{D^2}{4F} - C\right), \tag{23}$$

where $\sum_{k=1}^{n}\frac{-i}{x-x_k}$ is sum of the moving pole terms [26–33] that can be expressed as $\frac{P'(x)}{P(x)}$ where $P(x)$ is the polynomial:

$$P(x) = \prod_{k=1}^{k=n}(x - x_k). \tag{24}$$

Now the QMF in (20) can be expressed as

$$p(x, E) = -i\frac{P'(x)}{P(x)} + i\sqrt{F}x^2 + \frac{iD}{2\sqrt{F}}x - \frac{i}{2\sqrt{F}}\left(\frac{D^2}{4F} - C\right). \tag{25}$$

The differential equation for an arbitrary value of n, $P_n(x)$ can obtained by substituting QMF into the general equation (19). This differential equation has the following form:

$$\begin{aligned}P_n''(x) &- \left(2\sqrt{F}x^2 + \frac{D}{\sqrt{F}}x - \frac{D^2}{4F\sqrt{F}} + \frac{C}{\sqrt{F}}\right)P_n'(x)\\ &+ \left[E - A - \left(B + 2\sqrt{F} + \frac{D^3}{8F^2} - \frac{DC}{2F}\right)x\right.\\ &\left.- \frac{D^2C}{8F^2} + \frac{D^2}{64F^3} + \frac{C^2}{4F} - \frac{D}{2\sqrt{F}}\right]P_n(x) = 0.\end{aligned} \tag{26}$$

In order to solve this equation, we propose a polynomial solution for $P_n(x)$ that is given by

$$P_n(x) = \sum_{k=0} \lambda_k x^k, \tag{27}$$

where λ_k is a constant coefficient. Then, the wave function can be written as

$$\psi_n(x) \sim P_n(x)\exp\left[-\frac{\sqrt{F}}{3}x^3 - \frac{D}{4\sqrt{F}}x^2 + \frac{i}{2\sqrt{F}}\left(\frac{D}{4F} - C\right)x\right]. \tag{28}$$

It is known that these equations will have a non-trivial solution, let us investigate an explicit form of wave-function and energy eigenvalues for $n = 0$, 1 and 2.

For $n = 0$ case: In this case, QMF becomes

$$p(x, E) = i\sqrt{F}x^2 + \frac{iD}{2\sqrt{F}}x - \frac{1}{2\sqrt{F}}\left(\frac{D^2}{4F} - C\right) \tag{29}$$

and therefore the wave function is given by

$$\psi_0(x) = \exp\left(-\frac{\sqrt{F}}{3}x^3 - \frac{D}{4\sqrt{F}}x^2\right). \tag{30}$$

Then, equating the coefficients according to the powers of x in (27), one can obtain the energy eigenvalue as

$$E_0 = A + \sqrt{C}. \tag{31}$$

It is found that the relations among potential parameters D, F, C, and B can be written

$$D^2 = 4FC, \tag{32a}$$

$$B = -2\sqrt{F}. \tag{32b}$$

For $n = 1$ case: $P(x)$ is taken to be a first degree polynomial, $P(x) = x - x_0$ and from (26), x_0 is obtain as 0. Then we get

$$\psi_1(x) = Nx\exp\left(-\frac{\sqrt{F}}{3}x^3 - \frac{\sqrt{C}}{2}x^2\right), \tag{33}$$

where N is a normalization constant. The energy eigenvalue is given as

$$E_1 = A + 3\sqrt{C}. \tag{34}$$

For $n=2$ case: Here, $P(x)$ is taken as a second order degree polynomial. Substituting $P(x)$

$$P(x)=\lambda_0+\lambda_1 x+\lambda_2 x^2 \tag{35}$$

in (26), energy is obtained as

$$E_2^1=A+5\sqrt{C}, \qquad E_2^2=A+\sqrt{C}-2. \tag{36}$$

Therefore, the wave-functions are obtained for the energy spectrum in (36), respectively

$$\psi_2^1(x)=N_1\left(-\frac{C-4FC^2-16\sqrt{C}F^2}{16(2C\sqrt{F}-5F)+32F^2}x^2+1\right)\exp\left(-\frac{\sqrt{F}}{3}x^3-\frac{\sqrt{C}}{2}x^2\right); \tag{37a}$$

$$\psi_2^2(x)=N_2\left(-\frac{C-4FC^2-16\sqrt{C}F^2}{16F(4F-\sqrt{C}F+2C\sqrt{FC})}x^2+1\right)\exp\left(-\frac{\sqrt{F}}{3}x^3-\frac{\sqrt{C}}{2}x^2\right), \tag{37b}$$

where N_1, N_2 are normalization constants. These results agree with those obtained earlier QES quartic potential solution [31, 32].

4 Squeezing of Two-Mode Boson Hamiltonian

Squeezed boson states are defined through a new unitary transformation [43] $U=e^T$ with

$$T=\phi_A(a_1^+a_1^+)-\phi_A(a_1a_1)+\phi_B(a_2^+a_2^+)-\phi_B(a_2a_2), \tag{38}$$

where ϕ_A,ϕ_B are squeezed parameters. Squeezed boson states and the correlation between two modes of the bosons are defined through this transformation. The transformed operators take on the following forms:

$$U^{-1}a_1U=a_1\cosh 2\phi_A+a_1^+\sinh 2\phi_A, \tag{39a}$$

$$U^{-1}a_1^+a_1U=\sinh^2 2\phi_A, \tag{39b}$$

$$U^{-1}a_1a_1^+U=\cosh^2 2\phi_A, \tag{39c}$$

$$U^{-1}(a_1)^2U=\sinh 2\phi_A\cosh 2\phi_A. \tag{39d}$$

Similar relations hold for a_2 and a_2^+. Now we transform the Hamiltonian (11) by means of unitary transformation U to obtain squeezed boson Hamiltonian $\widetilde{H}=U^{-1}HU$. Then using (39) and taking $\phi_A=\phi_B=\phi$, we obtain

$$\begin{aligned}\widetilde{H}&=(\cosh 2\phi)^2(w_1a_1^+a_1+w_2a_2^+a_2)+\frac{1}{2}\sinh 4\phi((a_1^+)^2+(a_1)^2+(a_2^+)^2+(a_2)^2\\&\quad+\sinh 2\phi(a_1a_1^++a_2a_2^+)+a_1a_2(\alpha_2(\cosh 2\phi)^2+\alpha_1(\sinh 2\phi)^2)\\&\quad+a_1a_2\left(\alpha_1\frac{1}{2}\sinh 4\phi+\alpha_2\frac{1}{2}\sinh 4\phi\right)+a_1^+a_2^+(\alpha_2(\sinh 2\phi)^2+\alpha_1(\cos 2\phi)^2)\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\{a_1^2a_2^2(\cosh 2\phi)^4+a_1^2a_2a_2^+(\cosh 2\phi)^2\sinh 4\phi+a_1^2(a_2^+)^2(\cosh 2\phi)^2(\sinh 2\phi)^2\\
&+a_1a_1^+(a_2^+)^2(\sinh 2\phi)^2\sinh 4\phi+(a_1^+)^2a_2^2(\sinh 2\phi)^2(\cosh 2\phi)^2\\
&+a_2a_2^+(a_1^+)^2(\sinh 2\phi)^2\sinh 4\phi+(a_1^+)(a_2^+)^2(\sinh 2\phi)^4. \qquad (40)
\end{aligned}
$$

The transformed Hamiltonian can be expressed in terms of the generators of $SU(2)$ and $SU(1,1)$

$$
\begin{aligned}
\widetilde{H}=&(\cosh 2\phi)^2\left(w_1\frac{1}{2}(2K_0+L-1)+w_2\frac{1}{2}(2K_0-L-1)\right)\\
&+\sinh 2\phi\left\{\frac{1}{2}(2K_0+L-1)+\frac{1}{2}(2K_0-L-1)\right\}+K_-(\alpha_2(\cosh 2\phi)^2+\alpha_1(\sinh 2\phi)^2)\\
&+K_-\left(\alpha_1\frac{1}{2}(\sinh 4\phi)^2+\alpha_2\frac{1}{2}(\sinh 4\phi)^2\right)+K_+(\alpha_2(\sinh 2\phi)^2+\alpha_1(\cos 2\phi)^2)\\
&+\frac{1}{2}\sinh 4\phi(-4Q_2)-\frac{1}{2}\{K_-^2(\cosh 2\phi)^4+J_-^2(\cosh 2\phi)^2(\sinh 2\phi)^2\\
&+K_-K_+(\sinh 4\phi)^2+J_+^2(\sinh 2\phi)^2(\cosh 2\phi)^2+K_+^2(\sinh 2\phi)^4\\
&+K_-J_-\cosh(2\phi)^2\sinh 4\phi+K_-J_+\sinh 4\phi\cosh(2\phi)^2+J_-K_+(\sinh 2\phi)^2\sinh 4\phi\\
&+J_+K_+(\sinh 2\phi)^2\sinh 4\phi \qquad (41)
\end{aligned}
$$

and the transformed Hamiltonian can be cast into following differential form

$$
\begin{aligned}
\widetilde{H}=&-\frac{1}{2}\{[(\cosh 2\phi)^4+(\sinh 4\phi)^2(4+x^2+4x^4)+(\sinh 2\phi)^4x^4\\
&+(\cosh 2\phi)^2\sinh 4\phi(1-x^2)+(\sinh 2\phi)^2\sinh 4\phi(x^2-x^4)\\
&-2\sinh 4\phi(x^2-x^3)]\}\frac{d^2}{dx^2}+[(\cosh 2\phi)^2(w_1+w_2)x+2x\sinh 2\phi\\
&+\alpha_2(\cosh 2\phi)^2+\alpha_1(\sinh 2\phi)^2+\left(\frac{\alpha_1}{2}+\frac{\alpha_2}{2}\right)\sinh 4\phi+(\alpha_1(\cosh 2\phi)^2\\
&+\alpha_2(\sinh 2\phi)^2)x^2-\frac{1}{2}[(\sinh 4\phi)^2(2x+8x^3(1-j))\\
&+(\sinh 2\phi)^2\sinh 4\phi(2x-2x^2+2jx^3)]+\sinh 4\phi(-2x^2+jx^2-j-2kx^2-1)]\frac{d}{dx}\\
&+(\cosh 2\phi)^2(w_1(-2k-2)+w_22k-k(w_1+w_2))+\sinh 2\phi(-2k-1)\\
&-2kx(\alpha_2(\sinh 2\phi)^2+\alpha_1(\cosh 2\phi)^2)-\frac{1}{2}[(\sinh 4\phi)^2(-2k-8jx^2+16j^2x^2)\\
&+2jx(\cosh 2\phi)^2\sinh 4\phi+(\sinh 2\phi)^2\sinh 4\phi(-2k+2kx^2-4kjx^2)]+2kx\sinh 4\phi.
\end{aligned}
\tag{42}
$$

Now transformed Hamiltonian is rewritten

$$
\widetilde{H}=-\frac{1}{2}A(x)\frac{d^2}{dx^2}+B(x)\frac{d}{dx}+C(x), \tag{43}
$$

where

$$A(x)=\widetilde{a}_0+\widetilde{a}_1x^2+\widetilde{a}_2x^3+\widetilde{a}_3x^4, \tag{44a}$$
$$B(x)=\widetilde{b}_0+\widetilde{b}_1x+\widetilde{b}_2x^2+\widetilde{b}_3x^3, \tag{44b}$$
$$C(x)=\widetilde{c}_0+\widetilde{c}_1x+\widetilde{c}_2x^2 \tag{44c}$$

and

$$\widetilde{a}_0=4\sinh^2 4\phi+\cosh^4 2\phi+\cosh^2 2\phi\sinh 4\phi, \tag{45a}$$
$$\widetilde{a}_1=\sinh^2 4\phi-3\sinh 4\phi, \tag{45b}$$
$$\widetilde{a}_2=2\sinh 4\phi, \tag{45c}$$
$$\widetilde{a}_3=4\sinh^2 4\phi+\sinh^4 2\phi-\sinh^2 2\phi\sinh 4\phi, \tag{45d}$$
$$\widetilde{b}_0=\alpha_2\cosh^2 2\phi+\alpha_1\sinh^2 2\phi+\frac{\alpha_1}{2}\sinh 4\phi+\frac{\alpha_2}{2}\sinh 4\phi-\sinh 4\phi, \tag{45e}$$
$$\widetilde{b}_1=(\omega_1+\omega_2)\cosh^2 2\phi+2\sinh 2\phi-\sinh^2 4\phi-\sinh 4\phi, \tag{45f}$$
$$\widetilde{b}_2=\alpha_2\sinh^2 2\phi+\alpha_1\cosh^2 2\phi+(-2+j-2k)\sinh 4\phi, \tag{45g}$$
$$\widetilde{b}_3=(k-1)\sinh^4 2\phi-[(j-1)\sinh^2 2\phi-4(j-1)\sinh 4\phi]\sinh 4\phi, \tag{45h}$$
$$\begin{aligned}\widetilde{c}_0&=[-2\omega_1(k+1)+2k\omega_2-k(\omega_1+\omega_2)]\cosh^2 2\phi\\&\quad-(2k+1)\sinh 2\phi+k\sinh^2 4\phi+k\sinh^2 2\phi\sinh 4\phi,\end{aligned} \tag{45i}$$
$$\widetilde{c}_1=-2k(\alpha_2\sinh^2 2\phi+\alpha_1\cosh^2 2\phi)-j\cosh^2 2\phi\sinh 4\phi, \tag{45j}$$
$$\widetilde{c}_2=k(1-2j)\sinh^2 4\phi\sinh^2 2\phi-2j(4j+1)\sinh^2 4\phi-k(2k-1)\sinh^4 2\phi. \tag{45k}$$

This Hamiltonian can be written in the form of an eigenvalue equation

$$\widetilde{H}R(x)=ER(x), \tag{46}$$
$$-\frac{d^2R(x)}{dx^2}-W(x)\frac{dR(x)}{dx}+D(x)R(x)=F(x)R(x),$$

where $D(x)=\frac{2C(x)}{A(x)}$, $F(x)=\frac{2E}{A(x)}$ and

$$R(x)=e^{-\int W(x)dx}\psi(x). \tag{47}$$

With the wave function given in (47) the transformed Hamiltonian equation (46) can be transformed in the form of Schrödinger equation with the potential

$$V(x)=W'(x)+W(x)+D(x), \tag{48}$$
$$W(x)=-\frac{2(\widetilde{b}_0+\widetilde{b}_1x+\widetilde{b}_2x^2+\widetilde{b}_3x^3)}{\widetilde{a}_0+\widetilde{a}_1x^2+\widetilde{a}_2x^3+\widetilde{a}_3x^4}. \tag{49}$$

Substituting $W'(x)$, $W(x)$ and $D(x)$ in (48), the potential is obtained

$$V(x)=-\frac{2\sum_{i=0}^{i=7}\gamma_i x^i}{(\widetilde{a}_0+\widetilde{a}_1x^2+\widetilde{a}_2x^3+\widetilde{a}_3x^4)^2}, \tag{50}$$

where

$$\gamma_0 = -2\widetilde{c}_0\widetilde{a}_0 + \widetilde{b}_0\widetilde{a}_0 + \widetilde{b}_1\widetilde{a}_0, \tag{51a}$$

$$\gamma_1 = \widetilde{b}_1\widetilde{a}_0 - 2\widetilde{b}_0\widetilde{a}_1 + 2\widetilde{b}_2\widetilde{a}_0 - \widetilde{c}_1\widetilde{a}_0, \tag{51b}$$

$$\gamma_2 = -2\widetilde{c}_0\widetilde{a}_1 + \widetilde{b}_1\widetilde{a}_1 + 3\widetilde{b}_3\widetilde{a}_0 - \widetilde{c}_2\widetilde{a}_0 + \widetilde{b}_2\widetilde{a}_0 + \widetilde{b}_0\widetilde{a}_1 - 3\widetilde{b}_0\widetilde{a}_2, \tag{51c}$$

$$\gamma_3 = -\widetilde{c}_1\widetilde{a}_1 + \widetilde{b}_0\widetilde{a}_2 + \widetilde{b}_1\widetilde{a}_1 - 4\widetilde{b}_0\widetilde{a}_3 - 2\widetilde{b}_1\widetilde{a}_2 + \widetilde{b}_3\widetilde{a}_0 - 2\widetilde{c}_0\widetilde{a}_2, \tag{51d}$$

$$\gamma_4 = -\widetilde{b}_2\widetilde{a}_2 - \widetilde{c}_0\widetilde{a}_3 - \widetilde{c}_1\widetilde{a}_2 + \widetilde{b}_0\widetilde{a}_3 + \widetilde{b}_1\widetilde{a}_2 - 3\widetilde{b}_1\widetilde{a}_3 + \widetilde{b}_3\widetilde{a}_1 - \widetilde{c}_2\widetilde{a}_1 + \widetilde{b}_2\widetilde{a}_1, \tag{51e}$$

$$\gamma_5 = \widetilde{b}_3\widetilde{a}_1 - 2\widetilde{b}_2\widetilde{a}_3 + \widetilde{b}_2\widetilde{a}_2 - \widetilde{c}_2\widetilde{a}_2 + \widetilde{b}_1\widetilde{a}_3 - \widetilde{c}_1\widetilde{a}_3, \tag{51f}$$

$$\gamma_6 = \widetilde{b}_3\widetilde{a}_2 + \widetilde{b}_2\widetilde{a}_3 - \widetilde{c}_2\widetilde{a}_3 - \widetilde{b}_3\widetilde{a}_3, \tag{51g}$$

$$\gamma_7 = \widetilde{b}_3\widetilde{a}_3. \tag{51h}$$

If we introduce an ansatz for the wave function as

$$\psi(x) \sim \frac{x^n}{\widetilde{a}_1 x^2 + \widetilde{a}_2 x^3}, \tag{52}$$

where $n \leq 3$ and substitute (52) and (50) in the Schrödinger equation

$$\frac{d^2\psi}{dx^2} + (E - V(x))\psi(x) = 0, \tag{53}$$

the energy spectrum is obtained:

$$E_n = 2(2n+1) + \frac{2}{\widetilde{a}_0}(2\widetilde{c}_0 - \widetilde{b}_0 - \widetilde{b}_1). \tag{54}$$

By minimizing the last expression with respect to ϕ, we obtain one equation to solve. If we choose $\sinh 2\phi = t$ and $\alpha = k = j = w_1 = w_2 = 1$ for simplicity, t is found as -0.0608. Equations (17) and (50) are plotted in Fig. 1 as a function of x, where we also used the $\alpha = k = j = w_1 = w_2 = 1$ and $t = -0.0608$ for simplicity. As it is seen from the figure that the squeezed potential and effective potential have one minimum. When compared with

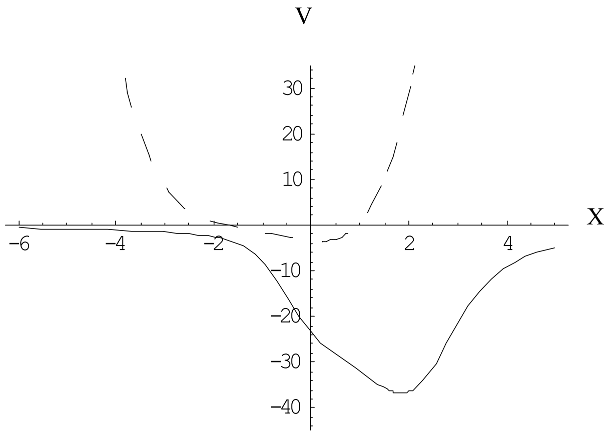

Fig. 1 The one dimensional potential representing the squeezed states (*full curve*) and the effective potential (*broken curve*) as a function

result of effective potential and squeezed potential, the depth of squeezed potential is greater than that of effective potential, thus energy appears to be reduced in the squeezed states. This justifies the general result that the squeezing effect gives a more stable state that of effective potential.

We have investigated squeezed Hamiltonian in the frame of group theoretical analyze. The group of transformed Hamiltonian equation (41) is different from Hamiltonian (12). Therefore it is found that squeezing acts as a symmetry breaking.

5 Conclusion

In this paper we have discussed the solution of the two-boson Hamiltonian. It was shown that the solution of this Hamiltonian can be transformed in the form of one-dimensional QES differential equation by applying suitable transformations. Quasi-exact solutions of this system are obtained by quantum Hamiltonian Jacobi formalism.

Then, we have constructed two-mode bosonic system through canonical transformation. The transformed Hamiltonian can be re-transformed in the form of one-dimensional QES differential equation and this equation is solved approximately. Then squeezed energy has been obtained for $n \leq 3$.

The method given here is useful to study nonlinear quantum optical systems. The range of the Hamiltonian can be extended by the Bogolibov transformation of the boson operators. It is expected that this work will leads to the solutions and construction of the multi-boson QES Hamiltonians.

References

1. Bender, C.M., Wang, Q.: J. Phys. A: Math. Gen. **34**, 9835–9847 (2001)
2. Bagchi, B., Gorain, P.S., Quesne, C.: Mod. Phys. Lett. A **21**, 2703–2708 (2006)
3. Rasinariu, C., Dykla, J.J., Gangopadhyaya, A., Mallow, J.V.: Phys. Lett. A **338**, 197–202 (2005)
4. Znojil, M.: J. Phys. A: Math. Gen. **39**, 4047–4061 (2006)
5. Mustafa, O., Mazharimousavi, S.H.: Phys. Lett. A **357**, 295–297 (2006)
6. Cannata, F., Ioffe, M.V., Nishnianidze, D.N.: Theor. Math. Phys. **148**, 960–967 (2006)
7. Cannata, F., Ioffe, M.V., Nishnianidze, D.N.: Teor. Mat. Fiz. **148**, 102–111 (2006)
8. Gomez-Ullate, D., Kamran, N., Milson, R.: arXiv:nlin.si/0401030
9. Kalnins, E.G., Miller, W. Jr., Pogosyan, G.S.: J. Math. Phys. **47**, 033502 (2006)
10. Turbiner, A.V.: Commun. Math. Phys. **118**, 467 (1988)
11. Razavy, M.: Am. J. Phys. **48**, 285 (1980)
12. Gangopadhyaya, A., Khare, A., Sukhatme, U.P.: Phys. Lett. A **208**, 261 (1995)
13. Tkachuk, V.M.: Phys. Lett. A **245**, 177 (1998)
14. Kuliy, T.V., Tkachuk, V.M.: J. Phys. A: Math. Gen. **32**, 2157 (1999)
15. Ushveridze, A.G.: Quasi-Exactly Solvable Models in Quantum Mechanics. Taylor and Francis, London (January 1, 1994), ISBN: 0750302666
16. Flugge, S.: Problems in Quantum Mechanics, vol. 1. Springer, Berlin (1971)
17. de Lange, O.L., Raab, R.E.: Operator Methods in Quantum Mechanics. Claredon, Oxford (1991)
18. Natanzon, G.A.: Theor. Math. Phys. **38**, 146 (1979)
19. Eisenhart, L.P.: Phys. Rev. **74**, 87 (1948)
20. Miller, W. Jr.: Symmetry and Separation of Variables. Addison-Wesley, Providence (1977)
21. Kalnins, E.G.: Separation of Variables for Riemannian Spaces of Constant Curvature. Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 28. Longman, Essex (1986)
22. Jordan, P.: Z. Phys. **38**, 513 (1926)
23. Schwinger, J.: Quantum Electrodynamics. Dover, New York (1958)
24. Leacock, R.A., Padgett, M.J.: Phys. Rev. Lett. **50**, 3 (1983)
25. Leacock, R.A., Padgett, M.J.: Phys. Rev. D **28**, 2491 (1983)
26. Bhalla, R.S., Kapoor, A.K., Panigrahi, P.K.: Am. J. Phys. **65**, 1187 (1997)

27. Bhalla, R.S., Kapoor, A.K., Panigrahi, P.K.: Mod. Phys. Lett. A **12**, 295 (1997)
28. Sree Ranjani, S., Kapoor, A.K., Panigrahi, P.K.: Mod. Phys. Lett. A **19**, 2047 (2004)
29. Sree Ranjani, S., Kapoor, A.K., Panigrahi, P.K.: Ann. Phys. (N.Y.) **320**, 164 (2005)
30. Sree Ranjani, S., Kapoor, A.K., Panigrahi, P.K.: Int. J. Theor. Phys. **44**, 1167 (2005)
31. Geojo, K.G., Sree Ranjani, S., Kapoor, A.K.: J. Phys A: Math. Gen. **36**, 4591 (2003)
32. Sree Ranjani, S., Geojo, K.G., Kapoor, A.K., Panigrahi, P.K.: Mod. Phys. Lett. A **19**(19), 1457 (2004)
33. Ranjani, S.S., Kapoor, A.K., Panigrahi, P.K.: Int. J. Mod. Phys. A **20**, 4067–4077 (2005)
34. Loudon, R., Knight, P.L.: J. Mod. Opt. **34**, 709 (1987)
35. Altanhan, T., Kandemir, B.S.: J. Phys. Condens. Matter **5**, 6729 (1993)
36. Altanhan, T., Kandemir, B.S.: J. Phys. Condens. Matter **9**, 427 (1997)
37. Lo, C.F., Wong, W.H.: Phys. Rev. B **52**, 3333 (1995)
38. Wodkiewics, K., Eberly, J.H.: J. Opt. Soc. Am. B **2**, 458 (1985)
39. Dolya, S.N., Zaslavskii, O.B.: J. Phys. A: Math. Gen. **33**, L369 (2000)
40. Dolya, S.N., Zaslavskii, O.B.: J. Phys. A: Math. Gen. **34**, 5955 (2001)
41. Alhassid, Y., Gürsey, F., Iachello, F.: Ann. Phys. **148**, 346 (1983)
42. Perelomov, A.: Generalized Coherent States and Their Applications. Springer, Berlin (1986)
43. Ocak, S.B.: Phys. Lett. A **335**, 402 (2005)